

**Static Solutions of the Einstein Equations
for Spherically Symmetric Elastic Bodies**

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Abstract

We analyze the Einstein equations for a spherically symmetric static distribution of matter which satisfies a given constitutive relation for elastomechanics. After reducing the equations into a system of Fuchsian ODE for certain scalar invariants of the strain, we show that, for a given constitutive relation and a given value of central pressure satisfying certain compatibility conditions, there exists a unique regular solution near the center. In the case when the constitutive relation is given by a quadratic form of strain, we further show that the solutions stay regular up to the boundary of the material ball.

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§1 Introduction

It is common in relativistic astrophysics to regard a star as a massive ball consisting of perfect fluid. Assuming that the ball is static, the corresponding Einstein equations reduce to the well known Tolman-Oppenheimer-Volkoff equation. Various authors have discussed properties of solutions of this equation. The existence theorem for this equation under general equation of state has been established by Rendall and Schmidt [1]. In the same article, as well as in a more recent article by Makino [2], the issue of finiteness of the ball and the regularity of solutions up to the boundary of the ball has been discussed.

Anisotropic material balls also have been objects of study and many particular solutions of the corresponding Einstein equations have been derived (for example [3], [4], [5] and [6]) under various assumptions imposed on the stress-energy tensor. But, unlike in the case of the perfect fluid ball, there hasn't been any general statement concerning the existence of solutions for anisotropic balls. The purpose of this paper is to resolve this problem for the elastic solids. To achieve this purpose, we use the relativistic elasticity theory formulated years ago by Kijowski and Magli [7]. Although there have been other complete theories of relativistic elastomechanics (for example [8] and [9]), the formalism of Kijowski and Magli seems to suit better for our purpose.

The main idea common in the modern theories of relativistic elasticity is that a material property is chosen by choosing a constitutive relation — the relation between the energy stored at a particle and the strain experienced by that particle. Once a constitutive relation has been chosen, the stress can be written as a function of strain and hence the material variables are replaced by the dynamic variables which describe the strain. Under this scheme, the Einstein equations for the spherically symmetric static elastic body can be written as a closed system of ODE for a variable describing the material configuration and two variables for the spacetime metric. In [7] some families of solutions of these equations have been produced, but no theorem regarding the existence of solutions of this system under general constitutive relations has been given.

In this paper, we show that for a given constitutive relation and a prescribed value of central pressure which satisfy certain compatibility conditions, there exists a unique smooth solution on a neighborhood of the center to the equations mentioned above. To prove the local existence and uniqueness, we use the theorem by Rendall and Schmidt [1] which states the existence, uniqueness and regularity of solutions for a certain class of Fuchsian ODE systems.

It is also shown that, in the case the given constitutive relation is a quadratic form of strain, the solutions can be extended to the boundary (where the radial stress vanishes), so that a Schwarzschild vacuum can be continuously joined outside

the ball. This is done by showing that the solutions are bounded on any finite open interval on which the radial stress stays positive. The most crucial step is to show that the mass-radius ratio is bounded from above by a constant $< \frac{1}{2}$. The similar issue actually arises in many other matter models under spherically symmetric setting. For perfect fluid and anisotropic fluid, this estimate has been made by Baumgarte and Rendall in [10] under the assumption that the tangential stress (in the case of anisotropic fluid) and the equation of state relating the energy density and the radial stress have been given and are sufficiently regular. In our paper the same estimate is made but in a different context; the regularity of tangential stress is not assumed a priori, but it is a consequence of the mass-radius ratio estimate. For Vlasov field, with isotropic or anisotropic stress, the same estimate has been established by G. Rein in [11]. We establish the bound of mass-radius ratio for the elastic material by adopting the method used by Rein in [11].

Throughout the paper, we use the spacetime metric signature $(-, +, +, +)$ and the units $c = G = 1$.

§2 A Brief Review of Relativistic Elasticity and Generalized Tolman-Oppenheimer-Volkoff Equations

In relativistic continuum mechanics, it is common to describe the configuration of particles by a map (*material map*) from a spacetime into a three dimensional Riemannian manifold (*material space*). This map $\xi : (M^4, g_{ij}) \rightarrow (X^3, \gamma_{ab})$ must satisfy the following conditions: (i) its differential $D\xi$ has full rank everywhere and (ii) at each $p \in M$, the solution $u^i \in T_p(M)$ of the equation $u^i \xi_{,i}^b = 0, b = 1, 2, 3$ is timelike. Both the projection $\eta_{ij} := g_{ij} + u_i u_j$ of the spacetime metric along u^i and the pull-back $(\xi^* \gamma)_{ij} := \xi_{,i}^a \xi_{,j}^b \gamma_{ab}$ of the material metric act as Riemannian metrics on the subspace of $T_p M$ which is orthogonal to u^i . When X represents an elastic body, the material metric γ_{ij} describes the infinitesimal distance between particles in a locally relaxed state. So, we say that an event $p \in M$ is at a relaxed state[†] if these two tensors agree on the orthogonal subspace in $T_p M$ of u^i . There are many different ways of describing the deviation of η_{ij} from $(\xi^* \gamma)_{ij}$ and any of them can be called a *relativistic strain* tensor. Among them we choose the following definition of strain as in [7].

$$S_i{}^j := -\frac{1}{2} \log \left((\xi^* \gamma)_i{}^j - u_i u^j \right) \quad (1)$$

Note that the operator to which “log” is applied is positive and self-adjoint, so the strain is well-defined by (1) for any choice of γ_{ab} and ξ . One can also show that this tensor is symmetric and is *spatial* in the sense that $S_{ij} u^j = 0$.

The main idea in the constitutive theory of relativistic elasticity is that the strain tensor should fully determine the elastic energy E stored at each particle, that is, E should be a function of S_{ij} . We call this function a *constitutive relation* for the material. Choosing this function $E = E(S_{ij})$ is equivalent to choosing a particular type of material to study.

The elastic energy E as a function of strain determines the stress-energy tensor which satisfies the conservation law $D_i T^i{}_j = 0$. The relationship between the stress-energy tensor and the strain tensor generalizes the *stress-strain relationship*; the details of its derivation can be found in [7]. We will quote the result for an isotropic elastic material only.

$$T_{ij} = n_o e^{-\alpha} \left[E u_i u_j - \frac{\partial E}{\partial \alpha} h_{ij} - \left(\frac{\partial E}{\partial \beta} \tilde{S}_{ij} + \frac{\partial E}{\partial \theta} \tilde{\tilde{S}} \tilde{\tilde{S}}_{ij} \right) \right] \quad (2)$$

[†] But, for a body under extreme pressure, a relaxed state might not exist. See [8] for a detailed discussion and an alternative formulation.

where “tilde” means “traceless part of”, n_o is the particle number density measured with respect to the pull-back metric $\xi^* \gamma_{ij}$ and

$$\alpha := \text{tr} S \quad (3)$$

$$\beta := \frac{1}{2} \text{tr} [\tilde{S} \cdot \tilde{S}] \quad (4)$$

$$\theta := \frac{1}{3} \text{tr} [\tilde{S} \cdot \tilde{S} \cdot \tilde{S}] \quad (5)$$

The quantity α measures the compression rate of the material relative to the relaxed state. More precisely, the particle number density n with respect to the projected spatial metric η_{ij} is given by

$$n = n_o e^{-\alpha}.$$

The energy density is therefore $\rho = nE = n_o e^{-\alpha} E$. The part between the round brackets in (2) (trace-free part) vanishes identically if E depends only on the compression α , i.e. when the material is a perfect fluid.

Now we will consider the case when the spacetime metric g_{ij} is spherically symmetric and static. We will assume the following form of spacetime metric.

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

Here the functions ν and λ are supposed to be smooth on M and depend only on the radial coordinate r . This is not the most general form of spherically symmetric spacetime metric since in general $e^{-\lambda}$ can vanish for some $r > 0$, but we will see in §4 that this cannot happen in the interior of material ball. We choose $X = \mathbf{R}^3$ as the material space and the Euclidian metric [†]

$$dy^2 + y^2 d\bar{\theta}^2 + y^2 \sin^2 \bar{\theta} d\bar{\phi}^2$$

as the material metric. We also assume that the particle distribution at the relaxed state is homogeneous, i.e. n_o is a constant. Under this assumption, we may set $n_o \equiv 1$ without loss of generality. A material map is given by specifying three functions $y(t, r, \theta, \phi)$, $\bar{\theta}(t, r, \theta, \phi)$ and $\bar{\phi}(t, r, \theta, \phi)$. But, by the assumption that the spacetime is static and spherically symmetric, y must depend on r only. After adjusting the angular coordinates for X if necessary, we may further assume that

[†] For more general material metrics, see Appendix.

$\bar{\theta} = \theta$ and $\bar{\phi} = \phi$. So, a material map is determined by specifying only one function $y : [0, \infty) \rightarrow [0, \infty)$. We will allow only smooth material maps, so $y(r) = rz(r^2)$ for some smooth function $z : \mathbf{R} \rightarrow \mathbf{R}$. We also insist that $y'(r) > 0$ for all $r \in [0, \infty)$ in order to ensure that the material map has full rank everywhere.

With a function $y(r)$ given for the material map, we can now form the strain tensor from (1). We have

$$S = -\log \frac{y'(r)}{\sqrt{e^{\lambda(r)}}} \left(e^{\lambda(r)} dr^2 \right) - \log \frac{y(r)}{r} (r^2 d\theta^2) - \log \frac{y(r)}{r} (r^2 \sin^2 \theta d\phi^2) \quad (6)$$

To write the stress-strain relationship (2), we need the partial derivatives of the energy with respect to the invariants of the strain. As a consequence of spherical symmetry, there are only two independent invariants of the strain tensor. Hence, the elastic energy E can be considered as a function of these two independent invariants of the strain. It turns out that the following choices of two invariants make the expression of the stress-energy tensor much simpler.

$$u := \log \left(\frac{ry'}{y\sqrt{e^\lambda}} \right) \quad (7)$$

$$v := \alpha + u = -3 \log \left(\frac{y}{r} \right) \quad (8)$$

The invariants α, β , and θ are related to u and v as follows.

$$\alpha = v - u, \quad \beta = \frac{1}{3}u^2 \quad \theta = -\frac{2}{27}u^3 \quad (9)$$

Using (2) and (9), we now get the stress-energy tensor.

$$T_{ij} = e^{u-v} \left[E e^\nu dt^2 + \frac{\partial E}{\partial u} e^\lambda dr^2 + \left(\frac{\partial E}{\partial u} - \frac{3}{2} \left(\frac{\partial E}{\partial u} + \frac{\partial E}{\partial v} \right) \right) r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad (10)$$

Let us pick notations for the components of the stress-energy tensor.

$$\rho := e^{u-v} E \quad (12)$$

$$P := e^{u-v} \frac{\partial E}{\partial u} \quad (13)$$

$$\Omega := \frac{1}{2} e^{u-v} \left(\frac{\partial E}{\partial u} + \frac{\partial E}{\partial v} \right) \quad (14)$$

$$Q := e^{u-v} \left(\frac{\partial E}{\partial u} - \frac{3}{2} \left(\frac{\partial E}{\partial u} + \frac{\partial E}{\partial v} \right) \right) \quad (15)$$

The Einstein equations can be now written.

$$8\pi r^2 \rho = e^{-\lambda}(r\lambda' - 1) + 1 \quad (16)$$

$$8\pi r^2 P = e^{-\lambda}(r\nu' + 1) - 1 \quad (17)$$

$$8\pi Q = \frac{1}{2}e^{-\lambda} \left(\nu'' + \frac{1}{2}\nu'^2 + \frac{1}{r}(\nu' - \lambda') - \frac{1}{2}\nu'\lambda' \right) \quad (18)$$

It should be remarked at this point that this is a closed system for three unknowns $\{y, \lambda, \nu\}$ since E is a function of u and v , which are again functions of y, y' , and e^λ .

Next, we will modify the Einstein equations (16)-(18) into the generalized Tolman-Oppenheimer-Volkoff equations by introducing a new variable

$$w := \frac{1}{2r^2} (1 - e^{-\lambda}) \quad (19)$$

With this new variable, the equation (16) becomes

$$rw' = 4\pi\rho - 3w \quad (20)$$

The other two equations become

$$P' = -\frac{6}{r}\Omega - r(1 - 2r^2w)^{-1} (4\pi P + w)(P + \rho) \quad (21)$$

and

$$\nu' = 2r(1 - 2r^2w)^{-1}(4\pi P + w) \quad (22)$$

The equation (22) can be integrated after all the other solutions have been found. So, we will focus on the equations (20) and (21) only.

We will assume throughout the paper that the constitutive function $E(u, v)$ is smooth. We will be also interested in only smooth solutions, that is, smooth material map and smooth metric coefficients. With these assumptions, it follows from (12)-(15) that all the variables which appear in the equations have to be smooth. So, we may consider them as smooth functions of $x = r^2$ defined on \mathbf{R} . The equations (20)-(21) then can be expressed using the new independent variable $x = r^2$ as follows.

$$2xP' = -6\Omega - x(1 - 2xw)^{-1}(4\pi P + w)(P + \rho) \quad (23)$$

$$2xw' = 4\pi\rho - 3w \quad (24)$$

Here, and throughout the rest of the paper, the “prime” indicates the derivation with respect to x . We also have an auxiliary equation from (7) and (8),

$$2xv' = 3 - 3(1 - 2xw)^{-\frac{1}{2}}e^u \quad (25)$$

The equations (23)-(25) form a closed system for the variables u , w , and v . If a set of smooth solutions $\{u(x), w(x), v(x)\}$ are found, then the solutions for the original equations will be determined algebraically or by a simple integration. The function y will be determined from (8) since it can be explicitly solved for y as $y(r) = re^{-v(r^2)}$. The metric variables λ and ν will be determined by (19) and (22) respectively [†]

[†] (19) can be solved for λ only if we know that $w < (2r^2)^{-1}$. It will be shown in §4 that indeed this inequality holds on any interval on which the radial stress P never vanishes.

§3 Local Existence and Uniqueness of Solutions

This section is concerned with the integrability of the equations (23)-(25) on a neighborhood of $x = 0$. After suitable substitutions and rearrangements we can write the equations explicitly in terms of u, w and v .

$$2xu' = \frac{3(E_u - E_{uv})(1 - \sqrt{b}e^u) - 3(E_u + E_v)}{(E_u + E_{uu})} - \frac{xb(4\pi e^{u-v}E_u + w)(E + E_u)}{E_u + E_{uu}} \quad (26)$$

$$2xw' = 4\pi e^{u-v}E - 3w \quad (27)$$

$$2xv' = 3 - 3\sqrt{b}e^u \quad (28)$$

where

$$b := (1 - 2xw)^{-1} \quad (29)$$

and E is a smooth function of u and v . These equations are singular at $x = 0$ and therefore the existence theorem for regular equations cannot be applied here. But, there exists a theorem which states existence, regularity and uniqueness of solutions for this type of equations, which we state below.

Theorem 1 (Rendall and Schmidt). *Let V be a finite dimensional vector space, $N : V \rightarrow V$ a linear map all of whose eigenvalues have positive real parts, and $G : V \times (-\epsilon, \epsilon) \rightarrow V$ and $g : (-\epsilon, \epsilon) \rightarrow V$ smooth maps, where $\epsilon > 0$. Then, there exists $\delta < \epsilon$ and a unique bounded C^1 function $f : (-\delta, 0) \cup (0, \delta) \rightarrow V$ which satisfies the equations*

$$x \frac{df}{dx} + Nf = xG(x, f(x)) + g(x) \quad (30)$$

Moreover, f extends to a smooth solution of (30) on $(-\delta, \delta)$. If N, G and g depend smoothly on a parameter t and the eigenvalues of N are distinct, then the solution depends smoothly on t .

Our equations are not in the form (30) yet. For instance, the equation (26) contains terms which are non-linear in u and v and those terms do not have a factor of x . One way to introduce a factor of x in those terms is to write $u = u_o + xu_1$ and $v = v_o + xv_1$ and to rewrite the equations for the new variables u_1, w and v_1 . By the way, (28) implies that $v(0)$ can be chosen freely and $u(0) = 0$ necessarily. So, we let

$$u = xu_1, \quad v = v_o + xv_1$$

Then the equations (26)-(28) become

$$2xu'_1 + 2u_1 = \frac{3(E_u - E_{uv})(1 - \sqrt{b}e^u) - 3(E_u + E_v)}{x(E_u + E_{uu})} - \frac{b(4\pi e^{u-v}E_u + w)(E + E_u)}{E_u + E_{uu}} \quad (31)$$

$$2xw' + 3w = 4\pi e^{u-v}E \quad (32)$$

$$2xv'_1 + 2v_1 = 3x^{-1}(1 - \sqrt{b}e^u) \quad (33)$$

We will now show that, if $E(u, v)$ and v_o satisfy certain conditions, then these equations can be rearranged into a system of the form given in Theorem 1. Let $U := (u_1, w, v_1)$.

Lemma 1. *Suppose (i) $E_u(0, v_o) + E_{uu}(0, v_o) \neq 0$ and (ii) $E_u(0, v) + E_v(0, v) = 0$ for all v . Then, the equations (31)-(33) can be rearranged into the form*

$$2xU' + \Lambda U = x\mathcal{G}(U, x) + V \quad (34)$$

Here $\mathcal{G} : \mathbf{R}^3 \times \mathbf{R} \rightarrow \mathbf{R}^3$ is a smooth map, V is a constant vector in \mathbf{R}^3 and

$$\Lambda = \begin{pmatrix} 5 & k & 0 \\ 0 & 3 & 0 \\ 3 & 3 & 2 \end{pmatrix},$$

where k is a constant which depends on v_0 .

Proof. Throughout the proof, “ $O(x)$ ” will represent a generic function of the form $xf(u_1, w, v_1, x)$ where f is a function which is smooth on a neighborhood of the set $x = 0$ in \mathbf{R}^4 .

(1) **The First Equation:** The first term on the right hand side of (31) is

$$3(E_u + E_{uu})^{-1} \left[(E_u - E_{uv})x^{-1} (1 - \sqrt{b}e^u) - x^{-1}(E_u + E_v) \right]$$

We have

$$\begin{aligned} x^{-1}(1 - \sqrt{b}e^u) &= x^{-1} \left[1 - (1 - 2xw)^{-\frac{1}{2}} e^{xu_1} \right] \\ &= x^{-1} \left[1 - (1 + xw)(1 + xu_1) + O(x^2) \right] \\ &= -w - u_1 + O(x) \end{aligned} \quad (35)$$

and

$$(E_u - E_{uv})(u, v) = (E_u - E_{uv})(0, v_o) + O(u) = (E_u - E_{uv})(0, v_o) + O(x)$$

So,

$$(E_u - E_{uv})x^{-1}(1 - \sqrt{b}e^u) = (E_u - E_{uv})(0, v_o) \cdot (-w - u_1) + O(x) \quad (36)$$

Next, using the second assumption, $(E_u + E_v)(0, v) \equiv 0$, we have

$$\begin{aligned} (E_u + E_v)(u, v) &= xu_1(E_u + E_v)_u(0, v_o) + xv_1(E_u + E_v)_v(0, v_o) + O(x^2) \\ &= xu_1(E_{uu} + E_{uv})(0, v_o) + O(x^2) \end{aligned}$$

So,

$$x^{-1}(E_u + E_v)(u, v) = u_1 \cdot (E_{uu} + E_{uv})(0, v_o) + O(x) \quad (37)$$

Combining (36) and (37), we get

$$\begin{aligned} (E_u - E_{uv})x^{-1}(1 - \sqrt{b}e^u) - x^{-1}(E_u + E_v) \\ &= (E_u - E_{uv})(0, v_o) \cdot (-u_1 - w) - (E_{uu} + E_{uv})(0, v_o) \cdot u_1 + O(x) \\ &= -(E_u + E_{uu})(0, v_o) \cdot u_1 - (E_u - E_{uv})(0, v_o) \cdot w + O(x) \end{aligned}$$

On the other hand, by the first assumption, $(E_u + E_{uu})^{-1}$ is a smooth at $(0, v_o)$. So, $(E_u + E_{uu})^{-1} = (E_u(0, v_o) + E_{uu}(0, v_o))^{-1} + O(x)$. Therefore,

$$\frac{3(E_u - E_{uv})(1 - \sqrt{b}e^u) - 3(E_u + E_v)}{x(E_u + E_{uu})} = -3u_1 - 3 \left(\frac{(E_u - E_{uv})(0, v_o)}{(E_u + E_{uu})(0, v_o)} \right) w + O(x) \quad (38)$$

The second term on the right hand side of (31) is of the form $b \cdot g(u, w, v)$, where g is a smooth function which is linear in w . Since $b = (1 - 2xw)^{-1} = 1 + O(x)$ we have

$$\begin{aligned} b \cdot g(u, w, v) \\ &= g(0, 0, v_o) + xu_1g_u(0, 0, v_o) + wg_w(0, 0, v_o) + xv_1g_v(0, 0, v_o) + O(x) \\ &= C_1 + C_2w + O(x) \end{aligned} \quad (39)$$

for some constants C_1 and C_2 which depend on v_o . Therefore, adding (38) with (39), the equation (31) becomes

$$2xu'_1 + 2u_1 = -3u_1 - kw + O(x) + V_1$$

or

$$2xu'_1 + 5u_1 + kw = O(x) + V_1$$

for some constant V_1 .

(2) **Second and Third Equations:** Since $e^{u-v}E(u, v)$ is smooth, we can write this as $C_1 + xu_1C_2 + xv_1C_3 + O(x) = C + O(x)$. So, the second equation is already in the desired form. The third equation has been already handled by (35).

[End of the Proof of Lemma 1]

Remark: The first hypothesis of this lemma restricts the choice of data v_o when a constitutive relation is given. But, unless $E_u + E_{uu} \equiv 0$ on the line $u = 0$, there will be always an open interval I such that the hypothesis is satisfied for all $v_o \in I$. This tells us that for almost all the choices of constitutive relation, there will be enough room to pick up v_0 with which the first hypothesis is satisfied.

A local existence theorem immediately follows from Theorem 1 and the above lemma.

Theorem 2. *Assume $E(u, v)$ and v_o satisfy the hypotheses of Lemma 1. Then there exists a positive constant δ such that the equations (27)-(28) admit a unique set of C^∞ solutions $\{u, w, v\}$ on the interval $(-\delta, \delta)$ satisfying $v(0) = v_o$.*

Suppose that the constitutive relation is given in the following form

$$E = m + \frac{A}{2}(u - v)^2 + \frac{B}{3}u^2 \quad (40)$$

where m represents the rest mass energy of a particle and A, B are nonnegative constants. Then, the condition (i) is satisfied for all v_o except for $v_o = 1 + \frac{2B}{3A}$ and the condition (ii) is also satisfied.

§4 Global Properties of Solutions for Quasi-Linear Models

In this section we study the global properties of solutions under the assumption that the constitutive relation is given by (40). The elasticity theory under such a form of constitutive relation is called the *quasi-linear elasticity* because under this constitutive relation the stress-strain relationship becomes linear. Under the quasi-linear constitutive relation of the form (40), the stress-energy tensor components (12)-(15) becomes

$$\rho := e^{u-v} \left[m + \frac{A}{2}(u-v)^2 + \frac{B}{3}u^2 \right] \quad (41)$$

$$P := e^{u-v} \left[A(u-v) + \frac{2B}{3}u \right] \quad (42)$$

$$\Omega := \frac{B}{3}ue^{u-v} \quad (43)$$

$$Q := e^{u-v} \left[A(u-v) - \frac{B}{3}u \right] \quad (44)$$

One of the properties of solutions in which we are interested is whether the radial stress P vanishes at some finite radial distance or not. If P vanishes at some finite radial distance $r = R$, we can consider the sphere $r = R$ as the boundary of the material body. If, further, the spacetime metric stays bounded inside this sphere, then a vacuum Schwarzschild solution may be attached outside the sphere to obtain a complete C^1 spacetime. Therefore another important question is whether the solution stays finite up to the radial distance where the radial stress vanishes.

This section focuses on answering the last question, but the first question will be answered in a special case when the anisotropy constant B is sufficiently close to 0. From now on we assume that $v(0) = v_o < 0$ is given. Since $u(0) = 0$, this means that we are assuming that the radial stress P at $x = 0$ is positive. We will show below that if P stays positive on a finite interval then all the variables u , v and w stay bounded and the equations (26)-(28) stay regular (except at $x = 0$) on the interval. The standard extension theorem of ODE then would imply that the interval of existence can be enlarged, i.e. the interval is not the maximal interval of existence yet.

Bounding w .

Let $[0, R)$, $R < \infty$, be an interval of existence of regular solutions of our equations (23)-(25) and assume $P > 0$ on the interval. Suppose w were not

bounded from above on the interval. Then by continuity of w on the interval, it would follow that $1 - 2x_1w(x_1) = 0$ for some $x_1 \in [0, R)$. Then from the equation (23), Ω would have to have a singularity at $x = x_1$. This contradicts the assumption that the solution is regular on $[0, R)$. Therefore w must be bounded from above. To obtain the lower bound, we use the equation (20) from which it follows that $w(0) = 4\pi\rho(0) > 0$ and

$$\frac{d}{dr}(r^3w(r)) = 4\pi\rho r^2 \quad (45)$$

for all r . So, $r^3w(r) \geq 0$ for all r . Therefore, $w \geq 0$. Moreover, because $\frac{d}{dr}(r^3w)$ is strictly positive (since $\rho > 0$), w must be bounded from below by a positive constant if we restrict the interval to $[x_o, R)$ for some $x_o > 0$.

Bounding e^λ .

First we will check whether the integral of ρ is bounded on finite intervals. From (45), we have $4\pi\rho(r) = r^{-2}\frac{d}{dr}(r^3w(r))$. Integrating this, we get

$$4\pi \int_0^r \rho(s)ds = rw(r) + 2 \int_0^r w(s)ds \quad (46)$$

So, it follows that the integral of ρ must be bounded on any finite interval since we know that w is bounded. Now, we are ready to show that e^λ is bounded. The proof of following lemma is based on an argument that has been used by G. Rein in [9].

Lemma 2. *Let $[0, R)$ be a finite interval on which the regular solution exists and $P > 0$. Then, e^λ is bounded on the interval.*

Proof. Using the original form of the Einstein equations (16) and (17) and the generalized Tolman-Oppenheimer-Volkoff equation (21) we can derive the following equation.

$$\left(e^{\frac{\lambda+\nu}{2}}(4\pi P + w) \right)' = 4\pi e^{\frac{\lambda+\nu}{2}} \left(\frac{\rho}{2x} - \frac{3w}{8\pi} - \frac{3\Omega}{x} \right) \quad (47)$$

On the other hand, from (41)-(44), it follows that for a choice of constant

$$k > \max \left(\sqrt{\frac{A}{m}}, \sqrt{\frac{3B}{2m}} \right)$$

we have

$$|P| < k\rho \quad \text{and} \quad |Q| < k\rho$$

and hence

$$|\Omega| \leq 2k\rho$$

So, noting that $w \geq 0$, we get the following inequality from the equation (47).

$$\left(e^{\frac{\lambda+\nu}{2}} (4\pi P + w) \right)' \leq K\rho \left(e^{\frac{\lambda+\nu}{2}} \right)$$

for some positive constant K . By restricting the interval to $[x_0, R)$ for some $x_0 > 0$ if necessary, we may assume that w is bounded from below by a positive constant. Then, the above inequality can be replaced by

$$\left(e^{\frac{\lambda+\nu}{2}} (4\pi P + w) \right)' \leq K\rho \left(e^{\frac{\lambda+\nu}{2}} (4\pi P + w) \right)$$

Applying Gronwall type argument to this inequality, we get

$$\left(e^{\frac{\lambda+\nu}{2}} (4\pi P + w) \right) (x) \leq C \exp \left(K \int_0^x \rho(s) ds \right)$$

for some positive constant C . So, using (46) and the earlier observation that w is bounded, we get

$$e^{\frac{\lambda+\nu}{2}} (4\pi P + w) \leq C$$

on $[0, R)$. From here, using again the fact that w is bounded from below by a positive constant we get $e^{\frac{\lambda+\nu}{2}} \leq C$. Now, the desired bound $e^{\frac{\lambda}{2}} \leq C$ follows since by the equation (22) ν is monotonically increasing.

[End of Proof]

Bounding the Rest.

As before, we assume that $P > 0$ on the interval $[0, R)$.

Step I. *Showing that v , P and Ω are bounded from above.* It follows immediately from the equation (25) that v is bounded from above. We get from (23)

$$2(xP)' = 2xP' - 2P \leq -6\Omega - 2P \leq 6|\Omega| + 2|P| \leq k \int_0^x \rho(s) ds$$

where, in the last step, we used the fact that $|P|$ and $|\Omega|$ are bounded by $k\rho$ for some constant k . This gives an upper bound of P since we know that the integral of ρ is bounded. The upper bound of Ω then follows from

$$2\Omega = P - A(u - v)e^{u-v} \leq P + 1$$

Step II. *Showing that u is bounded from above.* If $B > 0$, we have

$$ue^u = \frac{3}{B}\Omega e^v \leq C$$

So, u is bounded from above. If $B = 0$, then we know that $A > 0$, so we have

$$(u - v)e^{u-v} = \frac{P}{A} \leq C$$

Therefore $u - v$ is bounded from above. Since v is already bounded from above, it follows that u is bounded from above.

Step III. *Showing that u and v are bounded from below and the equation (26) remains regular within the interval $[0, R)$.* A lower bound of v comes from the equation (28) since we know that $\sqrt{b} := e^{\frac{\lambda}{2}}$ is bounded and that u is bounded from above. u is bounded from below since

$$e^{v-u}P = \left(A + \frac{2B}{3}\right)u - Av > 0$$

and v is bounded from below. The singularity of the equation (26) outside $x = 0$ can occur only when $E_u + E_{uu}$ vanishes. But, from (40),

$$E_u + E_{uu} = A(u - v) + \frac{2B}{3}u + A + \frac{2B}{3} = e^{v-u}P + A + \frac{2B}{3} > A + \frac{2B}{3} > 0$$

So, there is no singularity away from $x = 0$.

We have shown so far

Theorem 3. *Let $E(u, v)$ be given by (40). Given a constant $v_0 < 0$, let $[0, R)$ be the maximal interval on which the regular solution of (26)-(28) satisfying the initial condition $v(0) = v_0$ exists. Then P must vanish in the interval unless $R = \infty$.*

Remark: Note that this theorem applies only to the quasi-linear constitutive relation while the local result (Theorem 2) could be obtained for much larger class

of constitutive relations. Many of the arguments used to prove Theorem 3 rely on special features of the quasi-linear constitutive relation given by (40). But, the most crucial step — the boundedness of e^λ — could have been established for a much larger family of constitutive relations, namely those which satisfy the *dominant energy condition*. Note that under the dominant energy condition, the quantity $|\Omega|$ is bounded by 2ρ , which was the most important step in the proof of Lemma 2.

Finiteness of Radius

Suppose $B = 0$. Then $\Omega = 0$, so the material is an isotropic fluid. We will show that the radius is finite. Suppose $P > 0$ for all x . Then, by the monotonicity of P (see equation (23)), we know that P tends to a limit as $x \rightarrow \infty$. So, P' tends to 0 as $x \rightarrow \infty$. Then, again by the equation (23), it follows that $\rho \rightarrow 0$ and $P \rightarrow 0$ as $x \rightarrow \infty$. But

$$\rho = e^{u-v} \left(m + \frac{A}{2}(u-v)^2 \right) \quad \text{and} \quad P = Ae^{u-v}(u-v).$$

We can see from here that it is impossible for ρ and P both tend to 0 while $P > 0$. Therefore P must vanish somewhere. By the continuous dependence of solutions on parameters (Theorem 1), it immediately follows that for all sufficiently small B , the solutions must represent balls with finite radius.

§5 Discussion

We have shown that, given a smooth constitutive relation and a prescribed value of the stress at the center, there exists a unique static relativistic elastic ball. We also have shown, if the elastic energy is quadratic in the strain tensor, then either the radial stress vanishes at a finite radial distance before the solution becomes singular or the material fills up the whole space. If the radial stress vanishes at a finite radial distance $r = R$, then we can join a Schwarzschild vacuum outside the sphere $r = R$ and obtain a complete asymptotically flat C^1 solution of the Einstein equations. In the case that the radial stress never vanishes, which is certainly possible in theory, we can ask whether the solution represents an asymptotically flat spacetime or not, although it is hard to imagine an asymptotically flat spacetime filled with elastic material. In any case, it would be nice to be able

to tell, by looking at a given constitutive relation, whether the solution under that constitutive assumption represents an elastic ball of finite radius or infinite radius.

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Appendix

In §2, we have assumed that the material metric is flat. Since there is no reason to exclude non flat material space, we will consider the case with non-flat material metric. It turns out that the arguments used to prove the local existence and global properties in §3 and §4 applies to the case of non flat material space as well..

Because of the spherical symmetry, the most general form allowed for the material metric has the form

$$dy^2 + f^2(y) [d\bar{\theta}^2 + \sin^2 \bar{\theta} d\bar{\phi}^2]$$

where f is a non negative smooth function such that $f(y) = y \cdot g(y^2)$ for some smooth function g with $g(0) = 1$. We will impose an extra assumption that $f' \geq 0$ and g is a bounded function on \mathbf{R} . With this material metric, the strain tensor is

$$S = -\log \frac{y'(r)}{\sqrt{e^{\lambda(r)}}} \left(e^{\lambda(r)} dr^2 \right) - \log \frac{f(y)}{r} \left(r^2 d\theta^2 \right) - \log \frac{f(y)}{r} \left(r^2 \sin^2 \theta d\phi^2 \right)$$

Note that this expression can be obtained simply by replacing in (6) $y(r)$ by $f(y(r))$. As in §2, we proceed to choose the invariants u and v . This time we choose

$$u := \log \left(\frac{ry'}{f(y)\sqrt{e^\lambda}} \right) \tag{48}$$

$$v := \alpha + u = -3\log \left(\frac{f(y)}{r} \right) \tag{49}$$

Then we have the same stress-strain relationship (10) as in the case of flat material metric. Consequently the expressions in the the Einstein equations and the Tolman-Oppenheimer-Volkoff equation (12)-(24) are unchanged. The only deviation from the flat material case is that auxiliary equation (for v') is different.

$$2xv' = 3 - 3\sqrt{b}e^u \cdot f'(y) \tag{50}$$

Here we can consider y as an expression written in terms of x and v since f is invertible near $y = 0$.

We will show that the statement of Theorem 2 in §3 about the local existence of solutions is also true for the system (26),(27),(50). First, with the substitutions $u = xu_1$ and $v = v_o + xv_1$, we will show

$$x^{-1} \left(1 - \sqrt{b}e^u \cdot f'(y) \right) = -w - u_1 + O(x). \tag{51}$$

Once this has been shown, the local existence result for the system (26),(27),(50) follows immediately by following the proof of Lemma 1. (Recall that in the proof of Lemma 1 the auxiliary equation was used only for the verification of (35), which in the current case corresponds to (51).) The proof of (51) follows. From (49), $y = f^{-1}(re^{-\frac{v}{3}}) = O(r)$ as $r \rightarrow 0$. So, by the assumption on f , we have $f'(y) = g(y^2) + 2y^2 g'(y^2) = 1 + O(y^2) = 1 + O(r^2) = 1 + O(x)$. So, the left hand side of (51) can be treated as if there were no $f'(y)$ in the second term, which is the case in the proof of Lemma 1. Thus (51) has been verified and the local existence follows.

To obtain the global result stated in Theorem 3 in §4, we only need changes in the arguments leading to the existence of upper and lower bounds of v . To show that v is bounded from above, in §4, we have used the fact that the second term on the right hand side of (50) is positive. The same argument applies in the current case since we have assumed $f' \geq 0$. To show that v is bounded from below, it suffices to show that the second term on the right hand side of the equation (50) is bounded from above. The upper bound of u and $b := e^\lambda$ can be obtained independently of the equation (50) (See the proof of Lemma 2 and the Step II which follows this lemma.) So, it remains to show that $f'(y)$ is bounded from above. To show that $f'(y)$ is bounded, it suffices to show that y is bounded since f' is a continuous function. This will be done by using the equation (48). Since we know that e^λ and u are bounded from above, it follows from (48) that $y' \leq Kf(y)$ for some constant K . But, we have assumed that $f(y) = y g(y^2) \leq y \cdot C$ for some constant C . So, $y' \leq Ky$, and we have the bound for y .

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